J. Fluid Mech. (1994), *vol.* 276, *pp.* 189–210

Large-scale Langmuir circulation and double-diffusive convection: evolution equations and flow transitions

By STEPHEN M. COX[1] AND SIDNEY LEIBOVICH[2]

[1]Department of Applied Mathematics, The University of Adelaide, Adelaide 5005, Australia

[2]Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY, USA

(Received 16 February 1993 and in revised form 7 April 1994)

Two-dimensional Langmuir circulation in a layer of stably stratified water and the mathematically analogous problem of double-diffusive convection are studied with mixed boundary conditions. When the Biot numbers that occur in the mechanical boundary conditions are small and the destabilizing factors are large enough, the system will be unstable to perturbations of large horizontal length. The instability may be either direct or oscillatory depending on the control parameters. Evolution equations are derived here to account for both cases and for the transition between them. These evolution equations are not limited to small disturbances of the non-convective basic velocity and temperature fields, provided the spatial variations in the horizontal remain small. The direct bifurcation may be supercritical or subcritical, while in the case of oscillatory motions, stable finite-amplitude travelling waves emerge. At the transition, travelling waves, standing waves, and modulated travelling waves all are stable in sub-regimes.

1. Introduction

Langmuir circulation is a mechanically produced convective process of the surface layer of the ocean, and of lakes and ponds. The convection takes the form of rolls with axes generally aligned with the wind. The subsurface motion imprints the water surface with streaks, or 'windrows', parallel to the wind. This phenomenon, first observed by Irving Langmuir (1938), is believed to be an important contributor to the mixing of the upper layers of natural bodies of water (cf. the review by Leibovich 1983, and the experimental observations reported by Weller *et al.* 1985; Smith, Pinkel & Weller 1987; Weller & Price 1987). A theory by Craik & Leibovich (1976) attributes the convective mechanism to the distortion of wind-driven currents by the cumulative effects of wind-driven surface waves, and this has been elaborated by Craik (1977), Leibovich (1977*a, b*) and others.

The theory of the phenomenon, as well as casual observation, suggests a similarity with thermal convection and its generalization, double-diffusive convection, when the water is density stratified (usually stably so). Among other factors, the theory depends on the characteristics of the surface waves, which may be provided by supplying the wave spectrum. From this, the Stokes drift (Huang 1971) associated with the wave action can be calculated, and it is this that is needed as input to the theory. When the Stokes drift is approximated by a linear function of depth and the theory is restricted to motions independent of the wind direction, there is a strict mathematical analogy to double-diffusive convection in a fluid with unit Prandtl number. Although this analogy fails when the restriction is lifted, it permits the two-dimensional results a broader interpretation, including the oceanographically important case of thermohaline convection. We devote this paper to situations meeting this pair of idealizations, to permit a unified treatment of double-diffusive convection and Langmuir circulation.

Recently, two-dimensional Langmuir circulation and its physical relatives have been explored for a horizontally infinite layer of water of finite depth with a basic state consisting of rectilinear shear and linear temperature variation with depth. The recent work has assumed either Neumann (Leibovich, Lele & Moroz 1989; Cox *et al.* 1992*a, b*) or mixed (Cox & Leibovich 1993, referred to herein as CLI) boundary conditions applied to the perturbations to the basic state at the top and bottom surfaces of the layer of water. Neumann conditions were suggested as a model by Leibovich (1985). When the water layer is assumed infinite in the horizontal direction, Neumann conditions lead to an instability which has, at its critical threshold, a horizontally infinite wavelength. This feature of the instability is not an artifact of the idealization of constant Stokes drift gradient. The same feature arises in thermal convection (Sparrow, Goldstein & Jonsson 1964; Nield 1967), and also in Marangoni convection when the boundaries are assumed to be perfect thermal insulators.

This failure to yield a finite preferred wavelength led Sivashinsky (1982) to study Marangoni convection with mixed boundary conditions, and the same motive guided CLI in the Langmuir circulation problem. A physical argument for mixed boundary conditions in Langmuir circulation was given by CLI. The mixed boundary conditions are in the form $\partial\chi/\partial z + \beta\chi = 0$, where z is the vertical coordinate and χ is either the temperature (surrogate for buoyancy in the Boussinesq approximation) or the horizontal components of velocity, and the coupling coefficients represented schematically by β are sometimes called 'Biot numbers.' When the Biot numbers for the mechanical boundary conditions (which will be designated by the symbol α, with subscripts when appropriate for the top and bottom surface, at which different values are permitted) are very small in the Langmuir circulation problem, the horizontal lengthscales are large relative to the depth, the ratio being the aspect ratio of the system. An asymptotic development in the Biot numbers is then equivalent to one in inverse aspect ratio not dissimilar to shallow-water theory. This procedure can be used to obtain amplitude equations for the nonlinear motions. This has been carried out by CLI and CLII (where CLII refers to Cox & Leibovich 1994), generalizing the previous investigations of Chapman & Proctor (1980) for thermal convection and of Sivashinsky (1982) for Marangoni convection. Our amplitude equations are not restricted to weak nonlinearity, in the sense that the perturbations can be comparable to the variations of the field quantities in the basic state – provided the variations in the horizontal direction remain slow, and the underlying assumptions of the Craik–Leibovich theory (in the application to Langmuir circulation) are not violated.

CLI showed that when the ratio of the Biot number for the thermal boundary conditions to that for the mechanical boundary conditions is 'large', the onset of instability is determined by a (real) simple eigenvalue. Furthermore, attraction to the successor states of the system is determined by a single partial differential equation that is first order in time (Sparrow *et al.* 1964; Nield 1967; Chapman & Proctor 1980; Chapman, Childress & Proctor 1980; Gertsberg & Sivashinsky 1981; Depassier & Spiegel 1982; Sivashinsky 1982, 1983; Roberts 1985). The form of this amplitude equation follows that found by the previous investigators cited. Some of the properties and consequences of this equation are studied in our paper CLII.

Here we consider the case where the ratio of Biot numbers is 'small'. CLI showed that on linear grounds the bifurcation may be to steady or to oscillatory convection, but were unable to find a *nonlinear* counterpart to the asymptotic theory controlling post-bifurcation evolution of the system. Here this derivation is carried out. The resulting pair of nonlinear evolution equations inherits the O(2) symmetry (invariance with respect to horizontal translations and to reflection of the horizontal coordinates) of the underlying three-dimensional primitive equation system. A so-called Takens–Bogdanov bifurcation, where the linear operator has two zero eigenvalues, separates the cases of steady and oscillatory convection. Similar behaviours occur in thermohaline problems with stress-free isothermal, isohaline top and bottom boundaries. There, however, the bifurcations are degenerate and cannot be determined as easily as in the present case.

The steady bifurcation is supercritical for small values of the dimensionless stratification parameter S, but is subcritical for larger values of this parameter. Such behaviour is similar to that found with a mechanical Biot number of zero by Cox *et al.* (1992*a*).

The oscillatory bifurcation found here is always supercritical: travelling waves are predicted to be stable, while standing waves are unstable, again in agreement with Cox *et al.* (1992*a*).

At the Takens–Bogdanov bifurcation we find that the travelling-wave branch of solutions is at first stable, but transfers its stability to the standing wave branch through a branch of modulated travelling waves as the parameters change from their bifurcation values. All three kinds of waves should therefore be observable in numerical computations. The branch of steady solutions is subcritical and unstable. This behaviour differs from that found by Cox *et al.* (1992*a*) for the case of zero mechanical Biot number, where standing waves could not be stable.

After the first drafts of this paper had been completed, we became aware of a related analysis of a two-dimensional thermohaline problem by Cessi & Young (1992), and a referee has drawn our attention to two related papers by Hefer & Pismen (1987) and by Pismen (1988). Although each of these papers has connections with the subject and the analysis of this paper, they all differ in a number of respects. Cessi & Young (1992) consider the problem of forced thermohaline convection, driven by prescribing the salt flux and the temperature on the top surface, and zero fluxes of both temperature and salt at the lower boundary. Their analysis is based on a perturbation in aspect ratio, imposed *a priori* as a geometrical constraint through the forcing functions. By contrast, we consider an unforced stability problem, and are led to the introduction of large aspect ratio through the properties of the stability characteristics associated with boundaries which allow only small fluxes of temperature and salt. The pair of evolution equations found here permit oscillatory solutions as well as steady states, in contrast with the single evolution equation that controls the problem of Cessi & Young (1992), whose solutions are attracted to steady states. The papers by Hefer & Pismen (1987) and Pismen (1988) concern three-dimensional double-diffusive convection with boundary conditions like those treated in the present paper. Different equations are derived for each of the three cases – steady, oscillatory, and double-zero bifurcations – rather than the unified pair of equations that we obtain that encompass all of the separate cases. The analyses of Hefer & Pismen (1987) and Pismen (1988) apply asymptotically as the linear stability boundary is approached; and it is quite straightforward to show that our equations reduce to theirs in this limit. Pismen (1988) finds, with us, that travelling waves are preferred to standing waves near the oscillatory bifurcation in thermosolutal convection (but all are unstable to three-dimensional disturbances). Stability of two-dimensional solutions to three-dimensional perturbations in Langmuir circulation has not been investigated in this long-wave model. Pismen does not describe the solutions to be expected near the double-zero bifurcation.

2. Problem formulation

The mathematical problems for double-diffusive convection and for two-dimensional Langmuir circulation are very similar. By suitable assignments of parameters and suitable interpretation of the field variables, both can be embraced by a single mathematical model, to which our analysis will be applied. In this section, we lay out the two problems and their common model.

2.1. *The governing equations and boundary conditions for Langmuir circulation*

We consider the basic problem posed by CLI. In particular, we postulate a mixed layer bounded below by a strong thermocline. It is then plausible to suppose that vertical motions are inhibited by the thermocline, and that the vertical velocity can be neglected at the lower boundary of the layer. The layer is exposed to the atmosphere at the top boundary, and a constant wind speed exerts a stress on the air–sea interface, while at the same time surface waves propagate in the wind direction (taken to be in the direction of $x*$ increasing) and these waves are associated with a Stokes drift in the $x*$-direction with speed U_S. Here the asterisk denotes dimensional coordinates. The temperature of the air is also regarded as constant, and there is an associated heat transfer to the water that follows Newton's law of cooling, so that the heat flux is proportional to the difference between the air and the surface water temperatures. At the base of the mixed layer, heat transfer to the water below follows the same law of cooling, possibly with a different constant of proportionality. The stress there is assumed to be associated with entrainment or detrainment of quiescent waters below the layer by (relatively slow) deepening or thinning of the mixed layer. We start from the theoretical description of Langmuir circulation due to Craik & Leibovich (1976). The complete set of governing equations in the form needed is given in Leibovich (1977*b*), and consists of the wave-filtered Navier–Stokes equations and energy equations under the Boussinesq approximation.

The boundary conditions postulated permit a steady 'structureless equilibrium' to the wave-filtered equations of motion depending only on the depth. If $z*$ is the vertical coordinate ranging from $z* = -d$ at the mixed layer base to $z* = 0$ at the mean free surface, then the velocity of the structureless equilibrium is in the $x*$ direction with speed $U(z*)$ and the temperature is $T(z*)$, both depending linearly on $z*$. We then take $d\,\partial T/\partial z*$ as the scale for temperature, d as the unit for length, and d^2/ν_T as the unit for time, where ν_T is an eddy viscosity (assumed constant); dimensionless coordinates will be written without the asterisk. If the motion is assumed to be invariant in the x-direction, as we shall do, it proves convenient to scale the velocity component in the x-direction differently from those in the cross-wind plane (y, z). Accordingly, we take $d\,\partial U/\partial z*$ as the scale for the x-directed velocity component, and ν_T/d as the scale for the cross-wind velocity components. Since the motion is independent of x, we can adopt a streamfunction ψ to represent the dimensionless velocity components $v = \partial\psi/\partial z$ and $w = -\partial\psi/\partial y$. We let u be the dimensionless perturbation to the x-directed velocity component U, and we let θ be the dimensionless perturbation to the temperature T.

The governing equations for the dimensionless perturbations then are (Leibovich 1985)

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi = Rh(z)\frac{\partial u}{\partial y} - S\frac{\partial \theta}{\partial y} + J(\psi, \nabla^2\psi), \tag{2.1a}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)u = \frac{\partial \psi}{\partial y} + J(\psi, u), \tag{2.1b}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\theta = \frac{\partial \psi}{\partial y} + J(\psi, \theta). \tag{2.1c}$$

The parameters in these equations are

$$R = \frac{d^4}{\nu_T^2}\frac{\partial U}{\partial z}\frac{\partial U_S}{\partial z}(0), \quad S = \frac{\beta g d^4}{\nu_T^2}\frac{\partial T}{\partial z},$$

which represent the destabilizing vortex force and the stratification, respectively, where β is the coefficient of thermal expansion, g is the acceleration due to gravity, and

$$\tau = \frac{\kappa_T}{\nu_T}$$

is an inverse Prandtl number based on eddy coefficients of viscosity and thermal diffusivity. We assume throughout this paper that the mixed layer is either stably stratified or unstratified, that is, $S \geqslant 0$. The function $h(z)$ is the dimensionless Stokes-drift gradient, so

$$\frac{\partial U_S}{\partial z}(z) = \frac{\partial U_S}{\partial z}(0)h(z),$$

where $U_S(z)$ is the Stokes drift (in the $x*$ direction). Throughout this paper we shall take $h(z) \equiv 1$, which corresponds to making the Stokes drift a linear function of depth. The Jacobian is defined by $J(a, b) = a_y b_z - a_z b_y$.

The boundary conditions on the dimensionless perturbation temperature are

$$\frac{\partial \theta}{\partial z} + \gamma_t \theta = 0 \quad \text{on } z = 0, \tag{2.2}$$

and

$$\frac{\partial \theta}{\partial z} - \gamma_b \theta = 0 \quad \text{on } z = -1, \tag{2.3}$$

where γ_t and γ_b are small positive parameters. We define $\gamma = \gamma_t + \gamma_b$.

The boundary conditions on the perturbation velocity field are derived in CLI, and take the form

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2}\alpha_t \frac{\partial \psi}{\partial z} = \psi = \frac{\partial u}{\partial z} + \alpha_t u = 0 \quad \text{on } z = 0, \tag{2.4}$$

and

$$\frac{\partial^2 \psi}{\partial z^2} - \alpha_b \frac{\partial \psi}{\partial z} = \psi = \frac{\partial u}{\partial z} - \alpha_b u = 0 \quad \text{on } z = -1. \tag{2.5}$$

Here α_t and α_b are small positive parameters. We define $\alpha = \alpha_t + \alpha_b$.

The special case $\alpha_t = \alpha_b = \gamma_t^{-1} = \gamma_b^{-1} = 0$ has previously been the object of the several studies (Leibovich *et al.* 1989; Cox *et al.* 1992*a, b*) mentioned in the introduction. The derivation of the boundary conditions (2.2), (2.3), (2.4), (2.5), in particular the appearance of the factor 1/2 in the surface stress condition, is described in CLI.

Note that the problem is invariant under an arbitrary translation $y \to y + y_0$, and also under the transformation $(y, \psi) \to (-y, -\psi)$, properties that establish 'O(2) symmetry'. A physical consequence of this is that the system cannot distinguish left from right, so if travelling waves moving to the right are possible, then similar waves moving to the left are possible too.

CLI provide estimates for α_t, α_b and γ_t, γ_b for the Langmuir circulation problem. According to their estimates $\alpha_t \ll \alpha_b$, and the latter is estimated on the basis of the experiments of Kantha & Phillips (1976) to be

$$\alpha_b \sim 6 \times 10^{-3} \frac{U_a^2}{gd} R_*, \quad R_* = \frac{u_* d}{\nu_T},$$

where U_a is the wind speed, and u_* is the friction velocity in the water associated with the wind stress. If we contemplate a mixed-layer depth of 40 m and a wind speed of 10 m s^{-1}, then $\alpha_b \sim 8 \times 10^{-4} R_*$. A typical range for R_* is 5 to 50. According to CLI, the γ are expected to be comparable with, but perhaps smaller than, the α.

2.2. *The governing equations and boundary conditions for double diffusion*

The system (2.1) with $h(z) = 1$, describes thermohaline convection when the following interpretations are made: u is a temperature perturbation, θ a concentration, R a thermal Rayleigh number, S a solutal Rayleigh number, τ a Lewis number. In fact with this interpretation (2.1) holds only for a fluid with unit Prandtl number. For more general values of this parameter, which we denote by τ_p, the streamfunction ψ satisfies

$$\frac{1}{\tau_p}\left(\frac{\partial \nabla^2 \psi}{\partial t} - J(\psi, \nabla^2 \psi)\right) = R\frac{\partial u}{\partial y} - S\frac{\partial \theta}{\partial y} + \nabla^4 \psi.$$

The boundary conditions we have described are appropriate for almost-insulating horizontal boundaries with an almost-constant concentration flux. The momentum boundary conditions are the physically meaningful ones of constant stress (so that the perturbations are stress-free) if we replace the first conditions of (2.4) and (2.5) by $\partial^2 \psi / \partial z^2 = 0$ at each boundary. This difference between the boundary conditions appropriate to the Langmuir circulation and thermohaline convection problems turns out to be immaterial in the analysis that follows (the difference would be significant at a higher order than we compute in an asymptotic expansion).

The results we describe are phrased for the Langmuir circulation problem, but they apply with an appropriate interpretation to the thermohaline convection problem. In particular we give results for general values of τ_p– the Langmuir circulation problem corresponds to $\tau_p = 1$. We assume throughout that $\tau < 1$, except in §7, where we discuss the special case of $\tau = 1$.

3. When large-scale disturbances are important

The linear stability of the basic state may be examined by considering normal modes of the linearized governing equations, proportional to $e^{iky+\sigma t}$. When R is sufficiently small the basic state is stable to all small disturbances; as R is increased through a threshold value $R_c(k)$, dependent on the wavenumber k, the basic state becomes unstable to disturbances of that wavenumber. The first instability occurs when $R_c(k)$ is minimized. We denote this minimum value of R by R_c, and the corresponding wavenumber by k_c.

Figure 1 illustrates the marginal stability curves in four different circumstances.

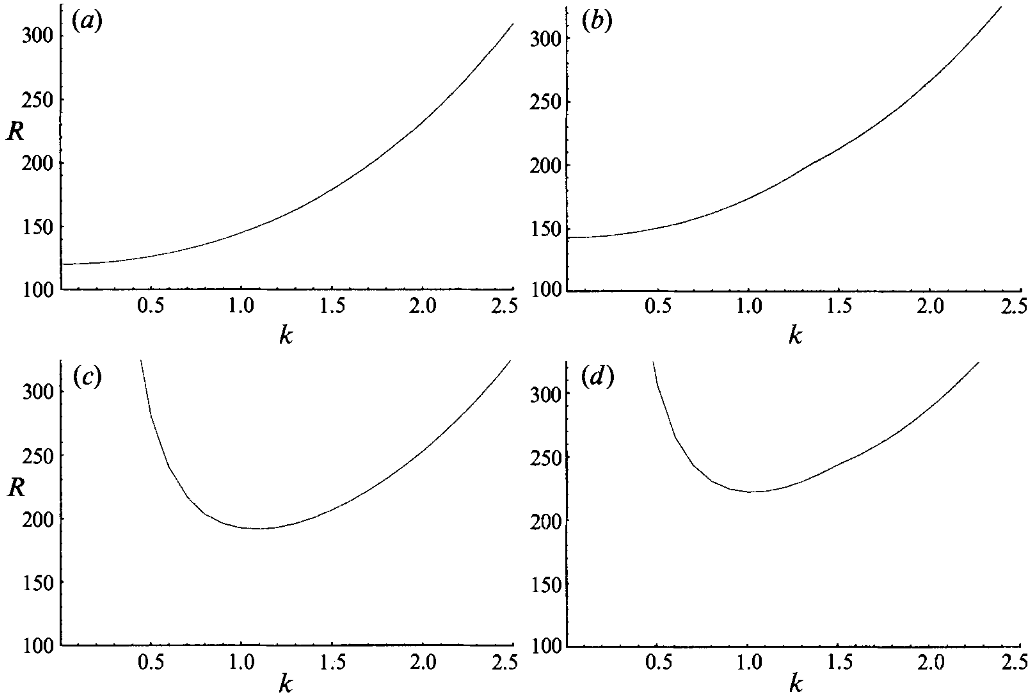

FIGURE 1. Some illustrative marginal stability curves for (2.1), computed numerically (see Cox *et al.* 1992*a* for more details): the critical value $R_c(k)$ is plotted against wavenumber, k. Below each curve the basic state is stable to small perturbations; it is unstable above. In all cases $\tau = 1/6.7$. (*a*) $\alpha_t = \alpha_b = 0$, $\gamma_t = \gamma_b = 0$, $S = 0$. (*b*) As for (*a*), except $S = 5$. (*c*) As for (*a*), except $\alpha_t = 0.06$, $\alpha_b = 0.24$. (*d*) As for (*a*), except $\alpha_t = 0.06$, $\alpha_b = 0.24$, $S = 5$. In cases (*a*) and (*b*), the critical wavenumber vanishes: $k_c = 0$ when $\alpha = 0$. In the other two cases $k_c > 0$ (when $\alpha > 0$). In cases (*a*) and (*c*) the bifurcation is steady (the eigenvalue $\sigma = 0$ at marginal stability) since the stratification of the mixed layer is weak ($S = 0 < S_d \approx 3.14$); in cases (*b*) and (*d*) the stronger stratification ($S = 5 > S_d$) leads to an oscillatory bifurcation ($\sigma = \pm i\omega$). (Small kinks in the marginal curves (*b*) and (*d*) indicate where the oscillatory bifurcation gives way to a steady bifurcation when k exceeds some value, around 1.3.)

When α vanishes, so does the critical wavenumber, k_c, and the linear instability occurs first with the largest horizontal scale available. (The infinite critical wavelength when flux boundary conditions are applied to u ($\alpha = 0$) is *not* a consequence of the assumption of constant Stokes-drift gradient. Other monotonic profiles for $h(z)$ yield $k_c = 0$.) When $\alpha > 0$, the critical wavenumber is positive, and so a finite wavelength is selected according to the linear theory. In fact $k_c = O(\alpha^{1/4})$ as $\alpha \to 0$, and this limit allows considerable analytical progress to be made, not only with the linear stability problem, but also with the derivation of governing equations for nonlinear disturbances.

It turns out that the value of γ is pivotal in deciding on the nature of the bifurcation. When $\gamma = O(1)$ (or $\gamma \gg 1$) the basic state becomes unstable in a steady bifurcation, as noted in the introduction. On the other hand, when γ is small (no larger in order of magnitude than α), a physically significant limit to consider, the basic state may become unstable to either steady or oscillatory convection, according to whether the stratification parameter S is small or large, respectively. This case of small γ is the subject of this paper.

Given the physical significance of disturbances of large horizontal lengthscale, we turn to an asymptotic analysis of this situation.

3.1. *Disturbances of extreme length*

For physically reasonable values of the parameters, in particular of α, the critical wavenumber k_c is small, and y-variations occur slowly. We begin by examining the limit obtained by ignoring y-variations altogether. In this case, both u and θ are determined by one-dimensional 'heat equations'

$$\frac{\partial u}{\partial t} - \mathrm{D}^2 u = 0, \qquad \frac{\partial \theta}{\partial t} - \tau \mathrm{D}^2 \theta = 0,$$

where $\mathrm{D} = \partial/\partial z$.

The solution to the initial-value problem is

$$u = \sum_{i=1}^{\infty} u_i(-\alpha_t \sin p_i z + p_i \cos p_i z)\mathrm{e}^{-p_i^2 t},$$

$$\theta = \sum_{i=1}^{\infty} \theta_i(-\gamma_t \sin q_i z + q_i \cos q_i z)\mathrm{e}^{-\tau q_i^2 t},$$

where the u_j, θ_j for $j = 1, 2, \ldots$ are constants and where p_i, q_i are determined by

$$(\alpha_t + \alpha_b)p_i = (p_i^2 - \alpha_t\alpha_b)\tan p_i,$$
$$(\gamma_t + \gamma_b)q_i = (q_i^2 - \gamma_t\gamma_b)\tan q_i.$$

The roots are ordered as follows: $0 \leqslant p_1 < p_2 < \ldots$, and $0 \leqslant q_1 < q_2 < \ldots$.

When α_t, α_b, γ_t, γ_b are all very small the components of the solution proportional to u_1 and θ_1 decay slowly – like $\exp(-\alpha t)$ and $\exp(-\tau\gamma t)$, where we recall our definitions

$$\alpha = \alpha_t + \alpha_b, \qquad \gamma = \gamma_t + \gamma_b.$$

The long-time motion is then determined by the slowly decaying contribution associated with p_1, q_1 – all other contributions decay more rapidly.

4. Long-wave approximation

Now we examine the evolution of the system when slow spatial dependence is permitted in the horizontal direction. To this end, and guided by the linear stability considerations discussed earlier, we write

$$\alpha_{t,b} = \epsilon^2 \bar{\alpha}_{t,b}, \quad \gamma_{t,b} = \epsilon^2 \bar{\gamma}_{t,b}, \quad Y = \epsilon^{\frac{1}{2}} y, \quad \psi = \epsilon^{\frac{1}{2}} \Psi.$$

With these substitutions, the governing equations take the form

$$(\mathrm{D}^2 - \partial_t)u + \epsilon\left[\partial_Y \Psi + J(\Psi, u) + \partial_Y^2 u\right] = 0,$$
$$(\tau\mathrm{D}^2 - \partial_t)\theta + \epsilon[\partial_Y \Psi + J(\Psi, \theta) + \tau\partial_Y^2 \theta] = 0,$$
$$\mathrm{D}^4\Psi - \partial_t \mathrm{D}^2\Psi/\tau_p + \partial_Y(Ru - S\theta) + \epsilon[J(\Psi, \mathrm{D}^2\Psi)/\tau_p$$
$$+2\mathrm{D}^2\partial_Y^2\Psi - \partial_t\partial_Y^2\Psi/\tau_p] + \epsilon^2[J(\Psi, \partial_Y^2\Psi)/\tau_p + \partial_Y^4\Psi] = 0.$$

Here the Jacobian J has been re-defined to involve derivatives with respect to Y rather than y. The boundary conditions are

$$\Psi = \mathrm{D}^2\Psi + \tfrac{1}{2}\epsilon^2\bar{\alpha}_t \mathrm{D}\Psi = \mathrm{D}u + \epsilon^2\bar{\alpha}_t u = D\theta + \epsilon^2\bar{\gamma}_t\theta = 0,$$

at $z = 0$, and

$$\Psi = D^2\Psi - \epsilon^2\bar{\alpha}_b D\Psi = Du - \epsilon^2\bar{\alpha}_b u = D\theta - \epsilon^2\bar{\gamma}_b\theta = 0,$$

at $z = -1$. These are the boundary conditions appropriate for Langmuir circulation (CLI), but for double-diffusive convection between stress-free boundaries the second of each set of conditions should be replaced by $D^2\Psi = 0$ at $z = 0, -1$.

We intend the parameter ϵ^2 to denote the order of magnitude of α and γ, so the parameters $\bar{\alpha}_{t,b}$ and $\bar{\gamma}_{t,b}$ are of order one. This leaves us some freedom in the choice of ϵ, and allows us easily to encompass the special case $\alpha = 0$. When $\alpha > 0$ we may for definiteness specify ϵ unambiguously by $\epsilon^2 = \alpha$, if we choose to do so. Note that although streamfunction perturbations are small ($O(\epsilon^{1/2})$), the total variation of the wind-directed velocity and the temperature are unrestricted by the smallness of ϵ: these variations may be arbitrarily large, provided variations taking place in the horizontal are *slow*.

We seek a solution asymptotically valid in the limit as $\epsilon \to 0$, and expand the unknowns as power series in the small parameter:

$$u = u_0 + \epsilon u_1 + \cdots, \quad \theta = \theta_0 + \epsilon\theta_1 + \cdots, \quad \Psi = \Psi_0 + \epsilon\Psi_1 + \cdots.$$

The coefficients of the powers of ϵ here are functions of Y, z and of time. To avoid secular terms and the consequent disordering of the series for large time, we must make provision for slow time variations. We choose to do this by the method of multiple timescales, supposing that all functions depend on the fast time $\bar{t} = t$, and slow times $T = \epsilon t$, $\hat{T} = \epsilon^2 t$, and so forth. Then we must also make the replacement

$$\partial_t = \partial_{\bar{t}} + \epsilon\partial_T + \epsilon^2\partial_{\hat{T}} + \cdots \tag{4.1}$$

in the governing equations.

Separating powers of ϵ, we generate a hierarchy of equations to solve in sequence. At the lowest order, we find u_0, θ_0 are independent of t and z, so

$$u = u_0(Y, T, \hat{T}),$$
$$\theta = \theta_0(Y, T, \hat{T}).$$

We subsequently *define* u_0 and θ_0 to be the depth-averaged windward velocity and temperature perturbations, respectively. The streamfunction to this order is

$$\Psi_0 = \frac{q(z)}{24}\partial_Y(S\theta_0 - Ru_0), \quad q(z) = z^4 + 2z^3 - z.$$

To leading order the shape of the streamlines does not depend on the nature of the convection (that is, steady or oscillatory).

At $O(\epsilon)$, the equations to be solved are

$$\left.\begin{aligned} (D^2 - \partial_{\bar{t}})\,u_1 &= -\partial_Y\Psi_0 - \partial_Y^2 u_0 - J(\Psi_0, u_0) + \partial_T u_0, \\ (\tau D^2 - \partial_{\bar{t}})\theta_1 &= -\partial_Y\Psi_0 - \tau\partial_Y^2\theta_0 - J(\Psi_0, \theta_0) + \partial_T\theta_0, \\ D^4\Psi_1 - \frac{\partial_{\bar{t}}D^2\Psi_1}{\tau_p} + \partial_Y(Ru_1 - S\theta_1) &= -\frac{J(\Psi_0, D^2\Psi_0)}{\tau_p} - 2D^2\partial_Y^2\Psi_0 + \frac{\partial_T D^2\Psi_0}{\tau_p}, \end{aligned}\right\} \tag{4.2}$$

subject to the boundary conditions

$$\Psi_1 = D^2\Psi_1 = Du_1 = D\theta_1 = 0, \quad \text{on } z = 0, -1.$$

The right-hand sides of (4.2) and (4.2) do not depend on $\bar{t}$. To avoid secular terms that depend linearly on the fast time, we must impose the orthogonality conditions

$$\int_{-1}^{0} \left(-\partial_Y \Psi_0 - \partial_Y^2 u_0 - J(\Psi_0, u_0) + \partial_T u_0\right) \mathrm{d}z = 0,$$
$$\int_{-1}^{0} \left(-\partial_Y \Psi_0 - \tau\partial_Y^2 \theta_0 - J(\Psi_0, \theta_0) + \partial_T \theta_0\right) \mathrm{d}z = 0.$$

These conditions lead to the evolution equations at lowest order

$$\left.\begin{aligned} \partial_T u_0 &= \partial_Y^2 \left(u_0 + \tfrac{1}{120}\left[S\theta_0 - Ru_0\right]\right), \\ \partial_T \theta_0 &= \partial_Y^2 \left(\tau\theta_0 + \tfrac{1}{120}\left[S\theta_0 - Ru_0\right]\right). \end{aligned}\right\} \tag{4.3}$$

Returning to the problem for u_1, θ_1, and making use of the orthogonality conditions (4.3) we find

$$u_1 = \frac{1}{120}\partial_Y^2(Ru_0 - S\theta_0)P(z) + \frac{1}{120}(R\partial_Y u_0 - S\partial_Y\theta_0)\partial_Y u_0 Q(z),$$
$$\theta_1 = \frac{1}{120\tau}\partial_Y^2(Ru_0 - S\theta_0)P(z) + \frac{1}{120\tau}(R\partial_Y u_0 - S\partial_Y\theta_0)\partial_Y \theta_0 Q(z).$$

where

$$P(z) = \frac{1}{56} - \frac{z^2}{2} - \frac{5z^3}{6} + \frac{z^5}{2} + \frac{z^6}{6},$$
$$Q(z) = -\frac{1}{2} + \frac{5z^2}{2} - \frac{5z^4}{2} - z^5,$$

Note that the constant terms in $P(z)$ and $Q(z)$ are chosen to make the depth average of u_1 and θ_1 vanish, in accord with our previous remarks.

The problem for Ψ_1 can now be solved. The solution is straightforward, but long, so we do not record it here.

Continuing to $O(\epsilon^2)$, we have the equations

$$(\mathrm{D}^2 - \partial_{\bar{t}})u_2 = \partial_{\hat{T}} u_0 - F_1 \tag{4.4}$$
$$(\tau\mathrm{D}^2 - \partial_{\bar{t}})\theta_2 = \partial_{\hat{T}} \theta_0 - F_2, \tag{4.5}$$

where

$$F_1 = \partial_Y \Psi_1 + \partial_Y^2 u_1 + J(\Psi_1, u_0) + J(\Psi_0, u_1) - \partial_T u_1,$$
$$F_2 = \partial_Y \Psi_1 + \tau\partial_Y^2 \theta_1 + J(\Psi_1, \theta_0) + J(\Psi_0, \theta_1) - \partial_T \theta_1,$$

subject to the boundary conditions

$$\left.\begin{aligned} &\mathrm{D}u_2 = -\bar{\alpha}_t u_0, &\quad \mathrm{D}\theta_2 = -\bar{\gamma}_t\theta_0 &\quad \text{at } z = 0, \\ &\mathrm{D}u_2 = \bar{\alpha}_b u_0, &\quad \mathrm{D}\theta_2 = \bar{\gamma}_b\theta_0 &\quad \text{at } z = -1. \end{aligned}\right\} \tag{4.6}$$

Integrating (4.4) and (4.5) across the layer and applying the boundary conditions (4.6), we find

$$-\bar{\alpha} u_0 = \partial_{\hat{T}} u_0 - \int_{-1}^{0} F_1 \, \mathrm{d}z,$$
$$-\tau\bar{\gamma}\theta_0 = \partial_{\hat{T}} \theta_0 - \int_{-1}^{0} F_2 \, \mathrm{d}z.$$

Carrying out the integrals and rearranging, we arrive at the results,

$$\left.\begin{aligned}\partial_{\hat{T}} u_0 &= -\bar{\alpha} u_0 - a_1 \partial_Y^4 u_0 + b_1 \partial_Y^4 \theta_0 + \tau c_1 \partial_Y \left[(R\partial_Y u_0 - S\partial_Y \theta_0)^2 \partial_Y u_0 \right], \\ \partial_{\hat{T}} \theta_0 &= -\tau\bar{\gamma} \theta_0 - a_1 \partial_Y^4 u_0 + b_1 \partial_Y^4 \theta_0 + c_1 \partial_Y \left[(R\partial_Y u_0 - S\partial_Y \theta_0)^2 \partial_Y \theta_0 \right],\end{aligned}\right\} \tag{4.7}$$

where

$$\begin{aligned} 79833600\tau a_1 &= R\left(31(S - R\tau) + 67320\tau(2\tau_p - 1)/\tau_p + 561(R - S)\tau/\tau_p\right), \\ 79833600\tau b_1 &= S\left(31(S - R\tau) + 67320\tau(2\tau_p - \tau)/\tau_p + 561(R - S)\tau/\tau_p\right), \\ 362880\tau c_1 &= 31. \end{aligned}$$

Now we combine the two slow timescales, invoking (4.1), (4.3), (4.7) and dropping the subscripts on u_0, θ_0 to give

$$\begin{aligned}\partial_t u = {}& -\epsilon^2 \bar{\alpha} u + \epsilon \partial_Y^2 \left[u + \frac{1}{120}(S\theta - Ru) \right] \\ & + \epsilon^2 \left\{ -a_1 \partial_Y^4 u + b_1 \partial_Y^4 \theta + \tau c_1 \partial_Y \left[(R\partial_Y u - S\partial_Y \theta)^2 \partial_Y u \right] \right\},\end{aligned}$$

$$\begin{aligned}\partial_t \theta = {}& -\epsilon^2 \tau \bar{\gamma} \theta + \epsilon \partial_Y^2 \left[\tau\theta + \frac{1}{120}(S\theta - Ru) \right] \\ & + \epsilon^2 \left\{ -a_1 \partial_Y^4 u + b_1 \partial_Y^4 \theta + c_1 \partial_Y \left[(R\partial_Y u - S\partial_Y \theta)^2 \partial_Y \theta \right] \right\}.\end{aligned}$$

If we restore the definitions $(\alpha, \gamma) = \epsilon^2(\bar{\alpha}, \bar{\gamma})$, and revert to the original horizontal variable $y = Y/\epsilon^{\frac{1}{2}}$, then the small parameter formally disappears from the pair of evolution equations. This neater form is

$$\begin{aligned}\frac{\partial u}{\partial t} = {}& -\alpha u + \frac{\partial^2}{\partial y^2}\left[u + \frac{1}{120}(S\theta - Ru) \right] \\ & + \left\{ -a_1 \frac{\partial^4 u}{\partial y^4} + b_1 \frac{\partial^4 \theta}{\partial y^4} + \tau c_1 \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\left(R\frac{\partial u}{\partial y} - S\frac{\partial \theta}{\partial y} \right)^2 \right] \right\},\end{aligned} \tag{4.8a}$$

$$\begin{aligned}\frac{\partial \theta}{\partial t} = {}& -\tau\gamma\theta + \frac{\partial^2}{\partial y^2}\left[\tau\theta + \frac{1}{120}(S\theta - Ru) \right] \\ & + \left\{ -a_1 \frac{\partial^4 u}{\partial y^4} + b_1 \frac{\partial^4 \theta}{\partial y^4} + c_1 \frac{\partial}{\partial y}\left[\frac{\partial \theta}{\partial y}\left(R\frac{\partial u}{\partial y} - S\frac{\partial \theta}{\partial y} \right)^2 \right] \right\}.\end{aligned} \tag{4.8b}$$

In the remainder of this paper we discuss the linear and weakly nonlinear stability of equations (4.8). Numerical solutions are described by Cox (1994).

5. Linear stability

We examine first the dynamics described by the linearized form of (4.8), and consider normal modes proportional to $e^{iky+\sigma t}$. The characteristic equation is the quadratic

$$\sigma^2 - (\mathscr{A} + \mathscr{D})\sigma + \mathscr{A}\mathscr{D} - \mathscr{B}\mathscr{C} = 0, \tag{5.1}$$

where

$$\mathscr{A} = -\alpha - \left(1 - \frac{R}{120}\right)k^2 - a_1 k^4,$$

$$\mathscr{B} = -\tau\Sigma k^2 + b_1 k^4,$$
$$\mathscr{C} = \frac{R}{120}k^2 - a_1 k^4,$$
$$\mathscr{D} = -\tau\gamma - \tau(1+\Sigma)k^2 + b_1 k^4,$$

and where for economy of symbols in later expressions we have introduced the notation

$$\Sigma = \frac{S}{120\tau}.$$

5.1. *Steady bifurcation*

The basic state becomes unstable to steady convection ($\sigma = 0$) when $\mathscr{A}\mathscr{D} - \mathscr{B}\mathscr{C} = 0$; the minimum of the marginal stability curve occurs when $\mathrm{d}(\mathscr{A}\mathscr{D} - \mathscr{B}\mathscr{C})/\mathrm{d}k = 0$ also. These conditions yield the critical value

$$R_c = 120\left\{1 + \Sigma + 2\left[(\alpha(1+\Sigma) - \gamma\Sigma)(a_1 - b_1/\tau)\right]^{1/2}\right\} + O(\epsilon^2), \tag{5.2}$$

and the critical wavenumber k_c, where

$$k_c^2 = \left[(\alpha(1+\Sigma) - \gamma\Sigma)/(a_1 - b_1/\tau)\right]^{1/2} + O(\epsilon^2).$$

In these expressions we evaluate a_1 and b_1 at $R = 120(1+\Sigma)$, and absorb the errors incurred by not using the exact value (5.2) of R_c in the terms of $O(\epsilon^2)$. In this way the expression for R_c is made explicit. In order for the square roots to be real, we require that $(\alpha(1+\Sigma) - \gamma\Sigma)(a_1 - b_1/\tau) > 0$. When we consider steady convection, we shall assume that this inequality is satisfied (otherwise our scaling assumptions fail to yield a non-zero value for the critical wavenumber). For a steady bifurcation, the coefficient of σ in the characteristic equation must be positive, that is,

$$120[\tau(1+\Sigma) + 1] - R + O(\epsilon^2) > 0.$$

This condition may be written (disregarding small terms) as $\Sigma < \Sigma_d$, where $\Sigma_d = \tau/(1-\tau) + O(\epsilon)$. Therefore if the stratification is sufficiently mild the bifurcation is predicted to be to steady convection.

5.2. *Oscillatory bifurcation*

An oscillatory bifurcation occurs when $\mathscr{A} + \mathscr{D} = 0$, that is, when

$$-\alpha - \tau\gamma + \left(\frac{R}{120} - [1 + \tau(1+\Sigma)]\right)k^2 - (a_1 - b_1)k^4 = 0.$$

Here we see that both α and γ act to stabilize the widest rolls, while the narrowest rolls are stabilized only if $a_1 - b_1 > 0$. We shall assume that this condition is satisfied (otherwise further terms in the small-k expansion – at least those of $O(\epsilon^3)$ – must be included to stabilize the smaller-scale motions (Knobloch 1989)). By solving $\mathscr{A} + \mathscr{D} = 0$ together with $\mathrm{d}(\mathscr{A} + \mathscr{D})/\mathrm{d}k = 0$ we find that the critical wavenumber is

$$k_c = \left(\frac{\alpha + \tau\gamma}{a_1 - b_1}\right)^{1/4},$$

and the critical value of R is

$$R_c = 120\left\{1 + \tau(1+\Sigma) + 2\left[(\alpha + \tau\gamma)(a_1 - b_1)\right]^{1/2}\right\}.$$

As for the analysis of the steady bifurcation, we make the expression for R_c explicit by evaluating a_1 and b_1 at $R = 120(1+\tau(1+\Sigma))$, and incur errors of $O(\epsilon^2)$.

At the onset of instability $\sigma = \pm ipk_c^2$, where

$$p^2 = \tau(1+\Sigma) - R_c\tau/120 + O(\epsilon) > 0.$$

This inequality may be rewritten as $\Sigma > \Sigma_d$, that is, for sufficiently strong stable stratification we expect an oscillatory bifurcation.

5.3. *Double-zero bifurcation*

If for some k and R both $\mathscr{A}+\mathscr{D}=0$ and $\mathscr{A}\mathscr{D}-\mathscr{B}\mathscr{C}=0$ then $\sigma=0$ is a double root of the characteristic equation (5.1). If, further, $\mathrm{d}(\mathscr{A}+\mathscr{D})/\mathrm{d}k=0$ and $\mathrm{d}(\mathscr{A}\mathscr{D}-\mathscr{B}\mathscr{C})/\mathrm{d}k=0$, then there is a pair of zero eigenvalues at the minimum of the marginal stability curve. To achieve this circumstance, the parameters must take values which we denote by $(R_d, \Sigma_d, \alpha_d, \gamma_d)$, and these together with the critical wavenumber k_c must satisfy

$$k_c = \left(\frac{\alpha_d + \tau\gamma_d}{d_1 - e_1}\right)^{1/4}, \qquad \Sigma_d = \frac{\tau}{1-\tau} + \frac{2e_1k_c^2}{\tau} + O(\epsilon^2),$$

$$R_d = 120(1+\tau(1+\Sigma_d)) + 240\left[(\alpha_d+\tau\gamma_d)(d_1-e_1)\right]^{1/2},$$

$$\gamma_d = \left(\frac{2182+561/\tau_p}{2182+561\tau/\tau_p}\right)\alpha_d + O(\epsilon),$$

where $(a_1, b_1) \to (d_1, e_1) + O(\epsilon)$ as $(R, \Sigma) \to (R_d, \Sigma_d)$, and

$$d_1 = \frac{1091+561\tau/\tau_p}{5544(1-\tau)}, \quad e_1 = \frac{(1091+561/\tau_p)\tau^2}{5544(1-\tau)}.$$

The regions of stability in (S, R)-space suggested by the analysis we have given above are given in figure 2. For small values of the stratification parameter S there is a steady bifurcation as R is increased, for larger values of S an oscillatory bifurcation, and in between is the double-zero point.

If we take molecular values for the viscosity ν and thermal diffusivity κ of water, we arrive at a value for τ of approximately 1/6.7. In this case $S_d = 120\tau\Sigma_d \approx 3.14$. Therefore for values of S larger than around 3 we expect an oscillatory bifurcation. If instead we take more realistic 'eddy' values for ν and κ then τ is nearer to 1. In the limit as $\tau \to 1-$ the value S_d dividing steady and oscillatory convection diverges, and steady convection is the most widely applicable case, except for extremely large values of the stratification parameter.

If γ is larger than $O(\epsilon^2)$ (for example, if $\gamma = O(1)$ or $\gamma = \infty$) then the bifurcation for small k is always steady, the temperature variable is zero at leading order, and a different analysis is required (CLI; CLII).

6. Weakly nonlinear analysis

6.1. *Steady convection*

Near the steady bifurcation of (4.8) we let the solution be expanded in powers of a small parameter, δ, so that

$$\left.\begin{aligned} u &= \delta U_1 + \delta^2 U_2 + \cdots, \\ \theta &= \delta\Theta_1 + \delta^2\Theta_2 + \cdots, \end{aligned}\right\} \tag{6.1}$$

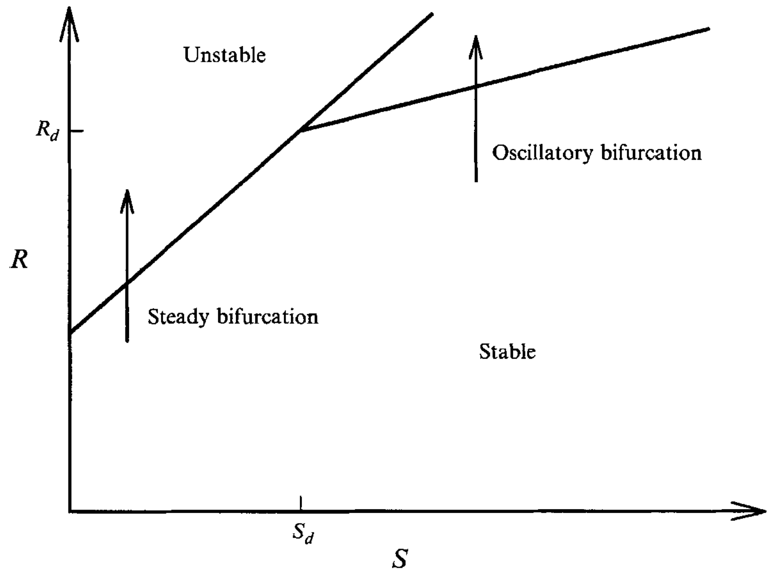

FIGURE 2. Stable and unstable regions in (S, R)-space (equivalently, in (Σ, R)-space, since $S = 120\tau\Sigma$). For small values of S (or Σ), increasing R destabilizes the basic state through a steady bifurcation; for larger values of S the bifurcation is oscillatory. At the point (S_d, R_d) the linear operator of (2.1) has a pair of zero eigenvalues – this is a Takens–Bogdanov bifurcation.

where $R = R_c + \delta^2 R_2$, and R_c is the critical value of R found in §5. Then to satisfy (4.8) at $O(\delta)$, we recover the linearized problem, and find

$$\begin{pmatrix} U_1 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \left(A(T)\mathrm{e}^{\mathrm{i}k_c y} + \text{c.c.} \right),$$

where $m = -\mathscr{A}/\mathscr{B} = 1/\tau + O(\epsilon)$. The slow timescale is now $T = \delta^2 t$, and c.c. denotes the complex conjugate of the preceding term. Because there are no quadratic nonlinear terms in (4.8) we set $U_2 = 0$.

The linearized operator at critical conditions is self-adjoint, so at $O(\delta^3)$ we form the inner product of (4.8) with the complex conjugate of the linear eigenvector, or more specifically with the vector $(\mathscr{D} \quad \mathscr{B})\,\mathrm{e}^{-\mathrm{i}k_c y}$. This leads to the Landau equation

$$(\mathscr{A} + \mathscr{D})\frac{\mathrm{d}A}{\mathrm{d}T} = (\mathscr{D} - \mathscr{B})\frac{R_2 k_c^2}{120}A - 3k_c^4 c_1 (R - mS)^2 (\mathscr{A} + \tau\mathscr{D})A|A|^2.$$

If we take only the leading-order contributions (in ϵ) to the coefficients we find that A satisfies

$$\frac{\Sigma_d - \Sigma}{\Sigma_d}\frac{\mathrm{d}A}{\mathrm{d}T} = k_c^2 \frac{R_2}{120}A - k_c^4 \frac{155(\Sigma^* - \Sigma)}{42\Sigma^*}A|A|^2, \tag{6.2}$$

where $0 < \Sigma^* = \tau^2/\left(1 - \tau^2\right) < \Sigma_d$. The steady bifurcation is therefore supercritical for $0 \leqslant \Sigma < \Sigma^*$, and subcritical for $\Sigma^* < \Sigma < \Sigma_d$. (Note that as $\tau \to 1-$, $\Sigma^*/\Sigma_d \to 1/2$, so that for half of its existence the steady bifurcation is supercritical, and for the other half subcritical.)

6.2. *Oscillatory convection*

For the oscillatory bifurcation, as for the steady bifurcation, we expand the variables u and θ, and assume that R is close to the critical value calculated in §5. Then to leading order in δ, the solution of (4.8) is the superposition of left- and right-travelling waves:

$$U_1 = \left(A_R(T)e^{i(k_c y-\omega t)} + A_L(T)e^{i(k_c y+\omega t)}\right) + \text{c.c.},$$
$$\Theta_1 = \left(mA_R(T)e^{i(k_c y-\omega t)} + m^* A_L(T)e^{i(k_c y+\omega t)}\right) + \text{c.c.},$$

where

$$\omega^2 = \mathscr{A}\mathscr{D} - \mathscr{B}\mathscr{C} = \tau^2\frac{\Sigma - \Sigma_d}{\Sigma_d}k_c^4 + O(\epsilon^3),$$

and now $m = -(i\omega + \mathscr{A})/\mathscr{B}$. Just as for the steady bifurcation, there is no quadratic forcing so we choose to set $U_2 = \Theta_2 = 0$.

At $O(\delta^3)$ in (4.8), and after much algebra we find

$$2i\omega\frac{dA_R}{dT} = \mu_1 A_R + \mu_2 A_R(|A_R|^2 + 2|A_L|^2), \tag{6.3}$$

where

$$\mu_1 = \frac{1}{120}k_c^2\left\{-k_c^2\tau + i\omega\right\}R_2 + O(\epsilon^3)$$

and

$$\mu_2 = 120c_1 k_c^4 R_c(1-\tau)\left\{-i\omega(1+\tau) + k_c^2\tau(\tau - 3 - \Sigma + \tau\Sigma)\right\} + O(\epsilon^4).$$

In what follows we ignore the small terms indicated in the expressions for μ_1, μ_2 only by their orders of magnitude. There is a similar governing equation for A_L, with the roles of A_R and A_L interchanged, and with complex-conjugated coefficients.

The analysis of the equations for A_R and A_L is standard. Below the marginal stability curve (that is, for $R_2 < 0$) the origin is the only fixed point, and is stable. Above the marginal curve, when $R_2 > 0$, there are four fixed points: the origin, with $A_R = A_L = 0$, corresponding to a state of no motion; a left-travelling wave with $A_R = 0$, $A_L = P_T e^{i\Omega T}$; a right-travelling wave with $A_L = 0$, $A_R = P_T e^{-i\Omega T}$; a standing wave with $|A_R| = |A_L| = P_T/\sqrt{3}$. Here

$$P_T = \frac{1}{120k_c}\left(\frac{R_2}{c_1 R_c(1-\tau^2)}\right)^{1/2},$$

and

$$\Omega = \frac{-\tau k_c^4 R_2(4 + (1-\tau)\Sigma)}{240\omega(1+\tau)}.$$

Since $\omega\Omega < 0$, the travelling waves decrease in speed as R_2 increases. The origin and the standing wave are unstable, while each travelling wave is stable. A bifurcation diagram is shown in figure 3.

In contrast to 'ideal double diffusion' (IDD), the stability of the travelling wave can be determined from this third-order analysis. For IDD the real part of the coefficient of $A_R|A_R|^2$ vanishes in (6.3), and fifth-order terms must be included to determine the stability of travelling waves.

6.3. *Double-zero bifurcation*

Near the double-zero bifurcation (Takens–Bogdanov bifurcation) at $(R, \Sigma) = (R_d, \Sigma_d)$, equations (4.8) may be written as

$$(u_0 - \tau\theta_0)_t = 0 + O(\epsilon^2),$$
$$(\tau u_0 - \theta_0)_t = (1+\tau)(u_0 - \tau\theta_0)_{yy} + O(\epsilon^2),$$

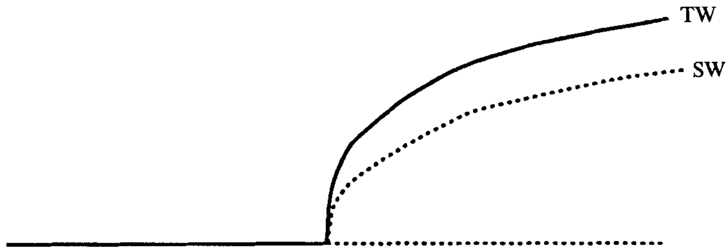

FIGURE 3. Bifurcation diagram for the oscillatory bifurcation. The parameter R increases from left to right, and the horizontal line represents the basic state. A solid line indicates a stable solution, while a dotted line indicates an unstable one. Both travelling waves (TW) and standing waves (SW) are supercritical; only TW are stable.

a form which clearly indicates proximity to a pair of zero eigenvalues.

When $k = k_c$, $R = R_d$ and $\Sigma = \Sigma_d$ the zero-eigenvector and generalized zero-eigenvector of (4.8) are, respectively,

$$\begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} \mathscr{B} \\ -\mathscr{A} \end{pmatrix} e^{ik_c y}, \qquad \begin{pmatrix} \mathscr{A}/(\mathscr{B}-\mathscr{C}) \\ \mathscr{B}/(\mathscr{B}-\mathscr{C}) \end{pmatrix} e^{ik_c y}.$$

Denoting the amplitudes of these vectors by P and Q, and substituting

$$\begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} \mathscr{B} \\ -\mathscr{A} \end{pmatrix} P(t)e^{ik_c y} + \begin{pmatrix} \mathscr{A}/(\mathscr{B}-\mathscr{C}) \\ \mathscr{B}/(\mathscr{B}-\mathscr{C}) \end{pmatrix} Q(t)e^{ik_c y}$$

into the linearized equations, we find

$$\frac{d}{dt}\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Now we turn to the nonlinear equations. As for the derivation of amplitude equations near the steady and oscillatory bifurcations, we expand u and θ as power series in the small parameter δ, in the form (6.1). To unfold the double-zero bifurcation at the fixed wavenumber k_c we must allow both R and Σ to vary independently:

$$R = R_d + \delta^2 R_2, \quad \Sigma = \Sigma_d + \delta^2 \Sigma_2.$$

In general, the ordinary differential evolution equations for P and Q at an O(2)-symmetric Takens–Bogdanov bifurcation may be written in the form (Dangelmayr & Knobloch 1987)

$$\left.\begin{aligned} P_t &= Q, \\ Q_t &= \delta^2 \left\{\mu P + \nu Q + \left[a|P|^2 + b|Q|^2 + c(PQ^* + P^*Q)\right] P + d|P|^2 Q\right\}. \end{aligned}\right\} \tag{6.4}$$

Here, μ and ν are unfolding parameters, linear combinations of R_2 and Σ_2. The Appendix describes the calculation of μ, ν and the coefficients a, c, d of the nonlinear terms (the coefficient b turns out not to be important in determining the bifurcations of (4.8)). Once we know these we refer to the catalogue of bifurcation diagrams given by Dangelmayr & Knobloch (1987) to select the one appropriate to the present problem. We use the approximate expressions that, to $O(\epsilon)$,

$$\mathscr{A} \sim \frac{\tau}{1-\tau}k_c^2, \quad \mathscr{B} \sim \frac{-\tau^2}{1-\tau}k_c^2, \quad \mathscr{C} \sim \frac{1}{1-\tau}k_c^2, \quad \mathscr{D} \sim \frac{-\tau}{1-\tau}k_c^2,$$

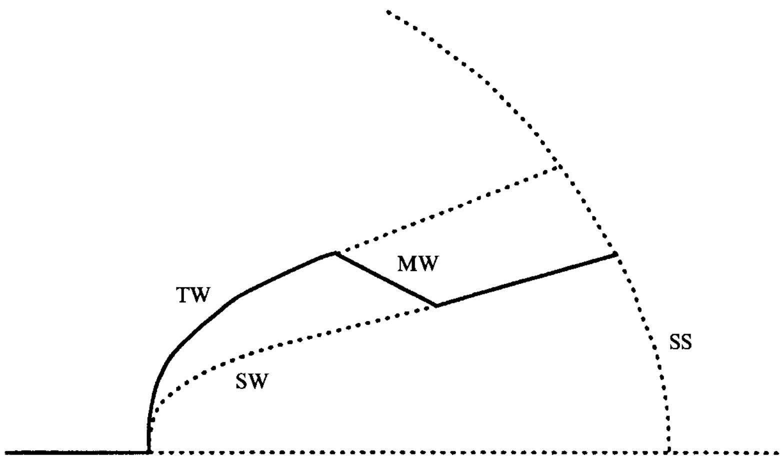

FIGURE 4. Bifurcation diagram for the double-zero bifurcation. Here $0 < \Sigma - \Sigma_d \ll \Sigma_d$, and R increases from left to right. Solid and dotted lines indicate stable and unstable solutions, respectively. The supercritical travelling waves (TW) are initially stable, but transfer their stability to the standing waves (SW) through a branch of stable modulated travelling waves (MW).

and find that the significant coefficients of the nonlinear terms in (6.4) are

$$a \sim 3f_1 k_c^{10}\tau^5 > 0,$$
$$c, d \sim -f_1 k_c^8 \tau^4 (1+\tau),$$

where $f_1 = 155/\left(126\tau(1-\tau)^2\right)$. An important quantity in determining the bifurcation structure of the solution is d/m, where $m = 2c + d$. Since $c = d$ then $d/m = 1/3$, which implies that we are in region '$III - (A > 0)$' of parameter space, in the notation of Dangelmayr & Knobloch (1987).

In order to describe the implications of our analysis in (S, R)-parameter space we need to relate the unfolding parameters μ, ν in (6.4) to the displacements from the critical point (S_d, R_d): the results in the Appendix give the relations

$$120\mu \sim k_c^4(\tau R_2 - S_2),$$
$$120\nu = k_c^2(R_2 - S_2).$$

According to Dangelmayr & Knobloch (1987) the interesting bifurcations occur in the region $\mu < 0$ as ν is increased through zero, which corresponds to increasing R with S just above S_d. The bifurcations of the system are indicated in figure 4. A sequence of bifurcations ensues as R increases between approximately $R_d + (S - S_d)$ and $R_d + (S - S_d)/\tau$: first there is a Hopf bifurcation which yields supercritical TW and SW with the former stable and the latter unstable. This is just as we have already discovered in our analysis of the oscillatory bifurcation. As R is further increased, however, the TW branch loses stability in a Hopf bifurcation as a branch of modulated waves (MW) bifurcates from it. This MW branch becomes heteroclinic to the SW branch after which the SW branch is stable. Upon further increase in R the SW branch becomes heteroclinic to the subcritical SS branch, and all the small-amplitude solutions predicted by (6.4) are unstable.

On the other hand, if S is just *below* S_d the only bifurcation as R is increased is the subcritical steady bifurcation we described earlier.

These qualitative results for the double-zero bifurcation are independent of τ_p, and so they apply directly to both Langmuir circulation and double-diffusive convection, with the boundary conditions we have indicated. For 'ideal double diffusion' the Takens–Bogdanov normal form is degenerate because the coefficient d vanishes, and the direction of bifurcation to TW cannot be decided until fifth-order terms are considered. Here, this degeneracy does not arise.

7. Equal eddy diffusivities for heat and momentum

If we set $\tau = 1$, so that the eddy diffusivities for heat and momentum are equal, then the equations (4.8) that govern u and θ are identical except for the terms derived from the boundary conditions. If $\alpha = \gamma$ then (4.8) may be collapsed to a single equation for the evolution of the 'total perturbation buoyancy' $B \equiv Ru - S\theta$ in (2.1):

$$\frac{\partial B}{\partial t} = -\alpha B + \left(1 - \frac{R-S}{120}\right)\frac{\partial^2 B}{\partial y^2} - \frac{a_1(R-S)}{R}\frac{\partial^4 B}{\partial y^4} + c_1 \frac{\partial}{\partial y}\left(\frac{\partial B}{\partial y}\right)^3 .$$

This is just the equation derived by Chapman & Proctor (1980), and by Gertsberg & Sivashinsky (1981) for Rayleigh–Bénard convection, in which there is a single diffusing species (temperature). This equation possesses a Lyapunov functional, which forces the long-time solution to approach a steady state.

On the other hand, if $\alpha \neq \gamma$, then (4.8) cannot be made into a single equation. The characteristic equation (5.1) permits only a steady bifurcation within its range of validity, when the Rayleigh number R takes the value

$$R_c = 120 + S + 2\left\{\frac{1091}{5544}\left[\alpha(1 + S/120) - \gamma S/120\right]\right\}^{1/2} + O(\epsilon^2).$$

The absence of an oscillatory bifurcation when $\tau = 1$ is consistent with the divergence of the threshold $\Sigma_d \sim \tau/(1-\tau)$ as $\tau \to 1$. The amplitude of steady convection satisfies a Landau equation which is the limit of (6.2) as $\tau \to 1$:

$$\frac{\mathrm{d}A}{\mathrm{d}T} = k_c^2 \frac{R_2}{120} A - k_c^4 \frac{155}{42} A|A|^2.$$

The bifurcation is always supercritical (the nonlinear term is stabilizing), indicating a smooth transition to motion.

8. Conclusions

In this paper we have shown that it is possible to find stable steady states, travelling waves, modulated waves and standing waves as post-bifurcation states in double-diffusive convection when the bounding horizontal surfaces are stress-free, with nearly constant heat flux, and nearly zero salt flux. The same set of possibilities has been shown for Langmuir circulation when the air/sea and mixed-layer/abyss boundaries have heat transfers that are only slightly affected by the differences in temperatures across the boundaries, and where the stresses transmitted across these boundaries are nearly constant.

Our analysis has proceeded from amplitude equations that we develop as asymptotic approximations based on the small departure of the perturbation fluxes from no-flux conditions, and the large aspect ratio of the motion that results from instabilities in such a system. The validity of the amplitude equations does not depend on weak nonlinearity. On the other hand, the amplitude equations lose validity if the coefficients a_1 and b_1 become very large. This can happen, for example, when the destabilizing Rayleigh number for the systems under consideration is highly supercritical. The stability threshold for most circumstances is reached when the Rayleigh number is of order 10^2; if it is an order of magnitude higher than this, a_1 and b_1 are large. We attribute this failure of the amplitude equations to the fact that shorter-wavelength instabilities are possible under highly supercritical conditions, and increases in a_1 and b_1, the coefficients of the highest-order spatial derivatives, are clearly the appropriate signal given by the approximation as notification of impending failure. In applications, the Rayleigh number can easily be large enough to render the approximation invalid. We note that the same limitation applies to the amplitude equation of Cessi & Young (1992).

S.M.C. is an Australian Research Council Australian Postdoctoral Research Fellow. The work of S.L. is supported by the National Science Foundation under grant OCE90-17882, and by the Office of Naval Research under grants N00014-92-J-1547 and the Marine Boundary Layer Accelerated Research Initiative.

Appendix. Computation of coefficients for the double-zero normal form

Here we record some of the steps in the algebra leading to the evaluation of the coefficients in (6.4). At $O(\delta)$ in the expansion for u and θ we let

$$\begin{pmatrix} U_1 \\ \Theta_1 \end{pmatrix} = \left[\begin{pmatrix} \mathscr{B} \\ -\mathscr{A} \end{pmatrix} P + \begin{pmatrix} \mathscr{A}/(\mathscr{B}-\mathscr{C}) \\ \mathscr{B}/(\mathscr{B}-\mathscr{C}) \end{pmatrix} Q \right] \mathrm{e}^{\mathrm{i}k_c y} + \mathrm{c.c.}$$

Then (4.8) is satisfied at $O(\delta)$. The absence of quadratic nonlinearity in (4.8) allows us to set $U_2 = \Theta_2 = 0$.

The terms of $O(\delta^3)$ in the governing equations provide solvability conditions on the coefficients in (6.4). For this purpose we need consider only the resonant terms at $O(\delta^3)$, that is, those proportional to $\mathrm{e}^{\mathrm{i}k_c y}$.

We write

$$\begin{aligned} U_3 = {} & [(R_2 v_1 + \Sigma_2 v_2)P + (R_2 v_3 + \Sigma_2 v_4)Q \\ & + P|P|^2 v_5 + Q|P|^2 v_6 + Q^* P^2 v_7 + P|Q|^2 v_8 + Q^2 P^* v_9 + Q|Q|^2 v_{10}]\, \mathrm{e}^{\mathrm{i}k_c y} \\ & + \mathrm{c.c.} + \cdots, \end{aligned}$$

where $v_1, \ldots, v_{10}$ are constants, and $\cdots$ represents terms proportional to $\mathrm{e}^{\pm 3\mathrm{i}k_c y}$ (non-resonant terms). By adding suitable multiples of the zero-eigenvector to U_3 we may set $\Theta_3 = 0$. Then at $O(\delta^3)$

$$\begin{aligned} U_t = {} & \big[(R_2 v_1 + \Sigma_2 v_2)Q + (2Q|P|^2 + P^2 Q^*)v_5 + (Q^2 P^* + P|Q|^2)v_6 \\ & + 2P|Q|^2 v_7 + Q|Q|^2 v_8 + Q|Q|^2 v_9 \\ & + \mathscr{A}\left\{\mu P + \nu Q + \left[a|P|^2 + b|Q|^2 + c(PQ^* + P^*Q)\right] P + d|P|^2 Q\right\}/(\mathscr{B}-\mathscr{C})\big]\, \mathrm{e}^{\mathrm{i}k_c y} \\ & + \mathrm{c.c.} + \cdots, \end{aligned}$$

where the first two lines arise from U_{3t} and the third from U_{1t}. Similarly,

$$\begin{aligned} \Theta_t = {} & \left[\mathscr{B}\left\{\mu P + \nu Q + \left[a|P|^2 + b|Q|^2 + c(PQ^* + P^*Q)\right] P + d|P|^2 Q\right\}/(\mathscr{B}-\mathscr{C})\right] \mathrm{e}^{\mathrm{i}k_c y} \\ & + \mathrm{c.c} + \cdots. \end{aligned}$$

At $O(\delta^3)$, considering only the resonant terms and omitting explicit reference to the factor $e^{ik_c y}$, we find the nonlinear term in the u-equation of (4.8) is

$$\begin{aligned}\mathcal{M}' \equiv -\tau f_1 k_c^4 \{ & 3\mathcal{B}(\mathcal{B}+\tau^2\mathcal{A})^2 P|P|^2 + 3\mathcal{A}(\mathcal{A}-\tau^2\mathcal{B})^2 Q|Q|^2/(\mathcal{B}-\mathcal{C})^3 \\ & + (\mathcal{B}+\tau^2\mathcal{A})(3\mathcal{A}\mathcal{B}+\tau^2(\mathcal{A}^2-2\mathcal{B}^2))(P^2Q^* + 2Q|P|^2)/(\mathcal{B}-\mathcal{C}) \\ & + (\mathcal{A}-\tau^2\mathcal{B})(3\mathcal{A}\mathcal{B}+\tau^2(2\mathcal{A}^2-\mathcal{B}^2))(Q^2P^* + 2P|Q|^2)/(\mathcal{B}-\mathcal{C})^2\},\end{aligned}$$

and in the θ-equation,

$$\begin{aligned}\mathcal{N}' \equiv -f_1 k_c^4 \{ & -3\mathcal{A}(\mathcal{B}+\tau^2\mathcal{A})^2 P|P|^2 + 3\mathcal{B}(\mathcal{A}-\tau^2\mathcal{B})^2 Q|Q|^2/(\mathcal{B}-\mathcal{C})^3 \\ & + (\mathcal{B}+\tau^2\mathcal{A})(3\tau^2\mathcal{A}\mathcal{B}+\mathcal{B}^2-2\mathcal{A}^2)(P^2Q^* + 2Q|P|^2)/(\mathcal{B}-\mathcal{C}) \\ & + (\mathcal{A}-\tau^2\mathcal{B})(3\tau^2\mathcal{A}\mathcal{B}+2\mathcal{B}^2-\mathcal{A}^2)(Q^2P^* + 2P|Q|^2)/(\mathcal{B}-\mathcal{C})^2\},\end{aligned}$$

where $f_1 = 155/(126\tau(1-\tau)^2)$. The linear terms in both the u and θ equations of (4.8) are

$$\frac{1}{120}R_2 k_c^2(\mathcal{B}P + \mathcal{A}Q/(\mathcal{B}-\mathcal{C})) - \tau\Sigma_2 k_c^2(-\mathcal{A}P + \mathcal{B}Q/(\mathcal{B}-\mathcal{C})) \equiv \tilde{\mathcal{M}}.$$

We define now

$$\mathcal{M} = \mathcal{M}' + \tilde{\mathcal{M}}, \quad \mathcal{N} = \mathcal{N}' + \tilde{\mathcal{M}}.$$

Then equations (4.8) become at $O(\delta^3)$

$$\begin{aligned} U_{1t} + U_{3t} &= \mathcal{A}(U_1+U_3) + \mathcal{B}\Theta_1 + \mathcal{M}, \\ \Theta_{1t} &= \mathcal{C}(U_1+U_3) + \mathcal{D}\Theta_1 + \mathcal{N}.\end{aligned}$$

Now we may proceed to solve a succession of equations to determine the coefficients in (6.4).

First consider the terms proportional to P at $O(\delta^3)$. These are

$$\begin{aligned} u : \frac{\mathcal{A}\mu}{\mathcal{B}-\mathcal{C}} &= \mathcal{A}(R_2 v_1 + \Sigma_2 v_2) + k_c^2(\frac{1}{120}R_2\mathcal{B} + \tau\Sigma_2\mathcal{A}), \\ \theta : \frac{\mathcal{B}\mu}{\mathcal{B}-\mathcal{C}} &= \mathcal{C}(R_2 v_1 + \Sigma_2 v_2) + k_c^2(\frac{1}{120}R_2\mathcal{B} + \tau\Sigma_2\mathcal{A}).\end{aligned}$$

From these it follows that

$$\begin{aligned} \mu &= k_c^2(\frac{1}{120}R_2(\mathcal{B}+\mathcal{A}) + \tau\Sigma_2(\mathcal{A}-\mathcal{C})), \\ v_1 &= k_c^2\frac{(\mathcal{A}-\mathcal{B})\mathcal{B}}{120\mathcal{A}(\mathcal{B}-\mathcal{C})}, \quad v_2 = k_c^2\tau\frac{\mathcal{A}-\mathcal{B}}{\mathcal{B}-\mathcal{C}}.\end{aligned}$$

Terms proportional to Q give

$$\begin{aligned} u : R_2 v_1 + \Sigma_2 v_2 + \frac{\mathcal{A}\nu}{\mathcal{B}-\mathcal{C}} &= \mathcal{A}(R_2 v_3 + \Sigma_2 v_4) + k_c^2(\frac{1}{120}\mathcal{A}R_2 - \mathcal{B}\tau\Sigma_2)/(\mathcal{B}-\mathcal{C}), \\ \theta : R_2 v_1 + \Sigma_2 v_2 + \frac{\mathcal{B}\nu}{\mathcal{B}-\mathcal{C}} &= \mathcal{C}(R_2 v_3 + \Sigma_2 v_4) + k_c^2(\frac{1}{120}\mathcal{A}R_2 - \mathcal{B}\tau\Sigma_2)/(\mathcal{B}-\mathcal{C}).\end{aligned}$$

The solution is

$$\nu = k_c^2(\frac{1}{120}R_2 - \tau\Sigma_2), \quad v_3 = k_c^2(\mathcal{B}-\mathcal{A})/(120\mathcal{C}(\mathcal{B}-\mathcal{C})), \quad v_4 = 0.$$

Turning now to the nonlinear terms in (4.8) at $O(\delta^3)$, we find those proportional to $P|P|^2$ to be

$$u : \frac{\mathscr{A}a}{\mathscr{B}-\mathscr{C}} = \mathscr{A}v_5 - 3\tau f_1 k_c^4 \mathscr{B}(\mathscr{B}+\tau^2\mathscr{A})^2,$$

$$\theta : \frac{\mathscr{B}a}{\mathscr{B}-\mathscr{C}} = \mathscr{C}v_5 + 3 f_1 k_c^4 \mathscr{A}(\mathscr{B}+\tau^2\mathscr{A})^2.$$

Solving for v_5 and a we obtain

$$a = 3f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})^2\mathscr{A}(1-\tau),$$

$$v_5 = 3f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})^2\mathscr{B}(\mathscr{C}-\tau\mathscr{B})/[\mathscr{A}(\mathscr{C}-\mathscr{B})].$$

Those terms proportional to $Q|P|^2$ are

$$u :$$

$$2v_5 + \mathscr{A}(c+d)/(\mathscr{B}-\mathscr{C}) = \mathscr{A}v_6 - \tau f_1 k_c^4(\mathscr{B}+\tau^2\mathscr{A})(6\mathscr{A}\mathscr{B}+\tau^2[2\mathscr{A}^2-4\mathscr{B}^2])/(\mathscr{B}-\mathscr{C}),$$

$$\theta : \mathscr{B}(c+d)/(\mathscr{B}-\mathscr{C}) = \mathscr{C}v_6 - f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})(6\tau^2\mathscr{A}\mathscr{B}+2\mathscr{B}^2-4\mathscr{A}^2)/(\mathscr{B}-\mathscr{C}).$$

The solutions for the unknowns are

$$c+d = -f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})(2\mathscr{B}+6\mathscr{B}\tau+6\mathscr{A}\tau^2+2\mathscr{A}\tau^3),$$

$$v_6 = f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})(2\mathscr{C}-3\mathscr{B}\tau-\mathscr{A}\tau^3)2\mathscr{B}/[\mathscr{C}(\mathscr{B}-\mathscr{C})].$$

The terms proportional to Q^*P^2 indicate that $c=d$, and so

$$c = d = -f_1k_c^4(\mathscr{B}+\tau^2\mathscr{A})(\mathscr{B}+3\mathscr{B}\tau+3\mathscr{A}\tau^2+\mathscr{A}\tau^3).$$

REFERENCES

Cessi, P. & Young, W. R. 1992 Multiple equilibria in two-dimensional thermohaline circulation. *J. Fluid Mech.* **241**, 291–309.

Chapman, C. J., Childress, S. & Proctor, M. R. E. 1980 Long wavelength thermal convection between non-conducting boundaries. *Earth & Planet. Sci. Lett.* **51**, 362–369.

Chapman, C. J. & Proctor, M. R. E. 1980 Nonlinear Rayleigh–Bénard convection between poorly conducting boundaries. *J. Fluid Mech.* **101**, 759–782.

Cox, S. M. 1994 Thermosolutal convection between poorly conducting plates. To appear in *J. Engng Maths.*

Cox, S. M. & Leibovich, S. 1993 Langmuir circulations in a surface layer bounded by a strong thermocline. *J. Phys. Oceanogr.* **23** 1330–1345 (referred to herein as CLI).

Cox, S. M. & Leibovich, S. 1994 Nonlinear scale selection in Langmuir circulation and in thermohaline convection. To be submitted (referred to herein as CLII).

Cox, S. M., Leibovich, S., Moroz, I. M. & Tandon, A. 1992*a* Nonlinear dynamics in Langmuir circulations with O(2) symmetry. *J. Fluid Mech.* **241**, 669–704.

Cox, S. M., Leibovich, S., Moroz, I. M. & Tandon, A. 1992*b* Hopf bifurcations in Langmuir circulations. *Physica* **59D**, 226–254.

Craik, A. D. D. 1977 The generation of Langmuir circulations by an instability mechanism. *J. Fluid Mech.* **81**, 209–223.

Craik, A. D. D. & Leibovich, S. 1976 A rational model for Langmuir circulations. *J. Fluid Mech.* **73**, 401–426.

Dangelmayr, G. & Knobloch, E. 1987 The Takens–Bogdanov bifurcation with O(2)-symmetry. *Phil. Trans. R. Soc. Lond.* A **322**, 243–279.

Depassier, M. C. & Spiegel, E. A. 1982 Convection with heat flux prescribed on the boundaries of the system. I. The effect of temperature dependence of material properties. *Geophys. Astrophys. Fluid Dyn.* **21**, 167–188.

GERTSBERG, V. L. & SIVASHINSKY, G. I. 1981 Large cells in nonlinear Rayleigh-Bénard convection. *Prog. Theor. Phys.* **66**, 1219–1229

HEFER, D. & PISMEN, L. M. 1987 Long-scale thermodiffusional convection. *Phys. Fluids* **30**, 2648–2654.

HUANG, N. E. 1971 Derivation of Stokes drift for a deep-water random gravity wave field. *Deep-Sea Res.* **18**, 255–259.

KANTHA, L. H. & PHILLIPS, O. M. 1976 On turbulent entrainment at a stable density interface. *J. Fluid Mech.* **79**, 753–768.

KNOBLOCH, E. 1989 Nonlinear binary fluid convection at positive separation ratios. In *Cooperative Dynamics in Complex Physical Systems* (ed. H. Takayama). Springer Series in Synergetics, vol. 43.

LANGMUIR, I. 1938 Surface motion of water induced by wind. *Science* **87**, 119–123.

LEIBOVICH, S. 1977*a* On the evolution of the system of wind drift currents and Langmuir circulations in the ocean. Part 1. Theory and averaged current. *J. Fluid Mech.* **79**, 715–743.

LEIBOVICH, S. 1977*b* Convective instability of stably stratified water in the ocean. *J. Fluid Mech.* **82**, 561–581.

LEIBOVICH, S. 1983 The form and dynamics of Langmuir circulations. *Ann. Rev. Fluid Mech.* **15**, 391–427.

LEIBOVICH, S. 1985 Dynamics of Langmuir circulations in a stratified ocean. In *The Ocean Surface* (ed. Y. Toba & H. Mitsuyasu), pp. 457–464. Reidel.

LEIBOVICH, S., LELE, S. & MOROZ, I. M. 1989 Nonlinear dynamics in Langmuir circulations and in thermosolutal convection. *J. Fluid Mech.* **198**, 471–511.

NIELD, D. A. 1967 The thermohaline Rayleigh–Jeffreys problem. *J. Fluid Mech.* **29**, 545–558.

PISMEN, L. M. 1988 Selection of long-scale oscillatory convective patterns. *Phys. Rev.* **38**, 2564–2572.

ROBERTS, A. J. 1985 An analysis of near-marginal, mildly penetrative convection with heat flux prescribed on the boundaries. *J. Fluid Mech.* **158**, 71–93.

SIVASHINSKY, G. I. 1982 Large cells in nonlinear Marangoni convection. *Physica* **4D**, 227–235.

SIVASHINSKY, G. I. 1983 On cellular instability in the solidification of a dilute binary alloy. *Physica* **8D**, 243–248.

SMITH, J., PINKEL, R. & WELLER, R. A. 1987 Velocity structure in the mixed layer during MILDEX. *J. Phys. Oceanogr.* **17**, 425–439.

SPARROW, E. M., GOLDSTEIN, R. J. & JONSSON, V. K. 1964 Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile. *J. Fluid Mech.* **18**, 513–528.

WELLER, R. A., DEAN, J. P., MARRA, J., PRICE, J. F., FRANCIS, E. A. & BOARDMAN, D. C. 1985 Three-dimensional flow in the upper ocean. *Science* **227**, 1552–1556.

WELLER, R. A. & PRICE, J. F. 1988 Langmuir circulation within the oceanic mixed layer. *Deep-Sea Res.* **35**, 711-747.